Period 1, Oct 21, 2024

~~16~~, 36, ~~12~~, 18, ~~8~~, ~~50~~, ~~20~~, ~~40~~, ~~46~~, ~~48~~, ~~24~~, 10

16. $x = \sec\frac{1}{y} = \sec(y^{-1})$

$\frac{d}{dx}\left[\frac{1}{y}\right] = \frac{d}{dx}\left[y^{-1}\right] = -1y^{-1-1}\cdot\frac{dy}{dx}$

$x = \sec u$

$\frac{-1}{y^2}\frac{dy}{dx}$

$\frac{dx}{dx} = 1 = \sec u \tan u \cdot \frac{du}{dx}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cos\frac{1}{y}\cdot -y^2 \cdot 1 = \cancel{\sec\frac{1}{y}}\cancel{\tan\frac{1}{y}}\cdot \frac{-1}{\cancel{y^2}}\frac{dy}{dx}\cdot \cancel{-y^2}\cdot\cancel{\cos\frac{1}{y}}\cdot\cancel{\cot\frac{1}{y}}$

$\cdot\cot\frac{1}{y}$

$-y^2\cos\frac{1}{y}\cot\frac{1}{y} = \frac{dy}{dx}$

10. $\underbrace{4\cos x}\,\underbrace{\sin y} = 1$

$4(-\sin x)\cdot\sin y + 4\cos x\cdot\cos y\cdot\frac{dy}{dx} = 0$

$-4\sin x\sin y + 4\cos x\cos y\frac{dy}{dx} = 0$

$+4\sin x\sin y \qquad +4\sin x\sin y$

$\frac{\cancel{4\cos x\cos y}\frac{dy}{dx}}{\cancel{4\cos x\cos y}} = \frac{4\sin x\sin y}{4\cos x\cos y}$

$\frac{dy}{dx} = \tan x\cdot\tan y$

12. $(\sin\pi x + \cos\pi y)^2 = 2$

$u = \sin\pi x + \cos\pi y$

$\frac{du}{dx} = \pi\cos\pi x + -\pi\sin\pi y\frac{dy}{dx}$

$y = \sin\pi x \Rightarrow y = \sin u$

Find $\frac{dy}{dx}$ $\qquad \frac{dy}{du} = \cos u$

$u = \pi x$

$\frac{du}{dx} = \pi \qquad \frac{du}{dx}\cdot\frac{dy}{du} = \frac{dy}{dx} = \pi\cdot\cos\pi x$

$u^2 = 2$

$2u\frac{du}{dx} = 0$

$0 = 2(\sin\pi x + \cos\pi y)(\pi\cos\pi x - \pi\sin\pi y\frac{dy}{dx})$

$0 = (2\sin\pi x + 2\cos\pi y)(\pi\cos\pi x - \pi\sin\pi y\frac{dy}{dx})$

$\pi(2\sin\pi x\cos\pi x) - \pi(2\sin\pi x\sin\pi y\frac{dy}{dx}) + 2\pi\cos\pi y\cos\pi x - \pi(2\sin\pi y\cos\pi y)\frac{dy}{dx}$

$0 = \pi\sin 2\pi x - 2\pi\sin\pi x\sin\pi y\frac{dy}{dx} + 2\pi\cos\pi y\cos\pi x - \pi\sin 2\pi y\frac{dy}{dx}$

$-\pi\sin 2\pi x$ $\quad$ $-2\pi\cos\pi y\cos\pi x$

$$\frac{-2\pi\cos\pi y\cos\pi x - \pi\sin 2\pi x}{(-2\pi\sin\pi x\sin\pi y - \pi\sin 2\pi y)} = \frac{dy}{dx}\frac{\cancel{(-2\pi\sin\pi x\sin\pi y - \pi\sin 2\pi y)}}{\cancel{(-2\pi\sin\pi x\sin\pi y - \pi\sin 2\pi y)}}$$

$$\frac{-2\pi\cos\pi y\cos\pi x - 2\pi\sin\pi x\cos\pi x}{-2\pi\sin\pi x\sin\pi y - 2\pi\sin\pi y\cos\pi y} = \frac{\cancel{-2\pi}\cos\pi x\cancel{(\cos\pi y + \sin\pi x)}}{\cancel{-2\pi}\sin\pi y\cancel{(\sin\pi x + \cos\pi y)}} = \frac{\cos\pi x}{\sin\pi y} = \frac{dy}{dx}$$

8. $\sqrt{xy} = x^2y + 1$ $\qquad$ $\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} = x^{\frac{1}{2}}\cdot y^{\frac{1}{2}}$

$x^{\frac{1}{2}}\cdot y^{\frac{1}{2}} = x^2 y + 1$

$\frac{1}{2}x^{\frac{1}{2}-1}\cdot y^{\frac{1}{2}} + x^{\frac{1}{2}}\cdot\frac{1}{2}y^{-\frac{1}{2}-1}\cdot\frac{dy}{dx} = 2x\cdot y + x^2\cdot\frac{dy}{dx} + 0$

$\cancel{\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}} + \frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}}\frac{dy}{dx} = 2xy + \cancel{x^2\frac{dy}{dx}} \Rightarrow \frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}}\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - \frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$

$-\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$ $\quad$ $-x^2\frac{dy}{dx}$ $\quad$ $-\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$ $\quad$ $-x^2\frac{dy}{dx}$

$$\frac{\frac{dy}{dx}\cancel{\left(\frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}} - x^2\right)}}{\cancel{\frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}} - x^2}} = \frac{2xy - \frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}}{\frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}} - x^2}$$

18, 20, 24

18 $x^2+y^2-4x+6y+9=0 \Rightarrow 2x+2y\frac{dy}{dx}-4+6\frac{dy}{dx}=0$

$-2x \quad +4 \qquad -2x+4$

$1x^2-4x+4 \quad +1y^2+6y+9 = -9+4+9$

$a=1$, $b=-4$, $\frac{b}{2}=\frac{-4}{2}=-2$, $\left(\frac{b}{2}\right)^2=(-2)^2=4$

$a=1$, $b=6$, $\frac{b}{2}=\frac{6}{2}=3$, $\left(\frac{b}{2}\right)^2=(3)^2=9$

$(2y+6)\frac{dy}{dx}=-2x+4$

$\frac{(2y+6)\frac{dy}{dx}}{(2y+6)}=\frac{-2x+4}{2y+6}$

$\frac{dy}{dx}=\frac{-2x+4}{2y+6}$

$(x-2)^2+(y+3)^2=4 \Rightarrow$ Circle center$(2,-3)$ $r=2$

$y=\pm\sqrt{-(x-2)^2+4}\;-3$

$\frac{dy}{dx}=\pm\frac{1}{2\sqrt{-(x-2)^2+4}}(2(x-2))=\frac{\pm2(x-2)}{2\sqrt{-(x-2)^2+4}}$

$\frac{dy}{dx}=\frac{-2(x-2)}{2\left(\pm\sqrt{(x-2)^2+4}-3\right)+6}$

$\frac{dy}{dx}=\frac{\pm2(x-2)}{2\sqrt{(x-2)^2+4}\;\cancel{-6+6}}$

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20. $16y^2-x^2=16 \Rightarrow 32y\frac{dy}{dx}-2x=0 \Rightarrow 32y\frac{dy}{dx}=2x \Rightarrow \frac{dy}{dx}=\frac{2x}{32y}$

$16y^2=16+x^2 \Rightarrow y=\pm\sqrt{\frac{16+x^2}{16}}=\frac{\pm\sqrt{16+x^2}}{\sqrt{16}}=\frac{\pm\sqrt{16+x^2}}{4}$

$32y\frac{dy}{dx}=2x$

$\frac{dy}{dx}=\frac{2x}{32y}=$

$y=\pm\left(\frac{(16+x^2)^{\frac{1}{2}}}{4}\right) \Rightarrow \frac{dy}{dx}=\frac{\pm1}{4}\cdot\frac{1}{2}(16+x^2)^{\frac{1}{2}-1}\cdot 2x = \frac{\pm x}{4\sqrt{(16+x^2)}}$

$$\frac{2x}{32\left(\frac{\pm\sqrt{16+x^2}}{4}\right)} = \frac{2x}{\pm 8\sqrt{16+x^2}} = \frac{\pm x}{4\sqrt{16+x^2}}$$

24.

$$(x+y)^3 = x^3+y^3 \quad (-1,1)$$

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2+3y^2\frac{dy}{dx}$$

$$3(-1+1)^2\left(1+\frac{dy}{dx}\right) = 3(-1)^2+3(1)^2\frac{dy}{dx}$$

$$\underbrace{3\cdot 0\left(1+\frac{dy}{dx}\right)} = 3+3\frac{dy}{dx}$$

$$0 = 3+3\frac{dy}{dx}$$

$$-3 \quad \cancel{-3}$$

$$-3 = 3\frac{dy}{dx} \Rightarrow -1 = \frac{dy}{dx}$$

36. $(\sqrt{3},1)$

$$7x^2 \underbrace{-6\sqrt{3}\,xy} + 13y^2 - 16 = 0$$

$$14x - 6\sqrt{3}\cdot y + -6\sqrt{3}x\cdot\frac{dy}{dx} + 13\cdot 2y\frac{dy}{dx} + 0 = 0$$

$$14x - 6\sqrt{3}y - 6\sqrt{3}x\frac{dy}{dx} + 26y\frac{dy}{dx} = 0$$

$$14\sqrt{3} - 6\sqrt{3}\cdot 1 - 6\sqrt{3}\cdot\sqrt{3}\frac{dy}{dx} + 26\cdot 1\frac{dy}{dx} = 0$$

$$\cancel{8\sqrt{3}} - \underbrace{6\cdot 3}_{18}\frac{dy}{dx} + 26\frac{dy}{dx} = 0$$

$$-8\sqrt{3} \qquad\qquad -8\sqrt{3}$$

$$\frac{\cancel{8}\frac{dy}{dx}}{\cancel{8}} = \frac{-8\sqrt{3}}{8}$$

$$\frac{dy}{dx} = -\sqrt{3}$$

$(\sqrt{3},1)$

$$y - y_1 = m(x-x_1)$$

$$y-1 = -\sqrt{3}(x-\sqrt{3})$$

$$y-1 = -\sqrt{3}x+3$$

$$+1 \qquad\qquad +1$$

$$y = -\sqrt{3}x+4$$

40, 46, 48, 50

40. $y^2(x^2+y^2) = 2x^2$ $\quad (1,1)$

$$2y\frac{dy}{dx}(x^2+y^2) + y^2\left(2x+2y\frac{dy}{dx}\right) = 4x$$

$$2\cdot 1\cdot\frac{dy}{dx}(1^2+1^2) + 1^2\left(2\cdot 1 + 2\cdot 1\cdot\frac{dy}{dx}\right) = 4\cdot 1$$

$$\underbrace{2\cdot 2}_{4}\frac{dy}{dx} + \cancel{2} + 2\frac{dy}{dx} = 4$$

$$-2 \qquad -2$$

$$\Rightarrow 6\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{3}$$

$$y-1 = \frac{1}{3}(x-1)$$

$$y-1 = \frac{1}{3}x - \frac{1}{3}$$

$$+1 \qquad\qquad +1$$

$$y = \frac{1}{3}x + \frac{2}{3}$$

46.

$$x^2y^2 - 2x = 3$$

$$\cancel{2x \cdot y^2} + x^2 \cdot 2y\frac{dy}{dx} - \cancel{2} = 0$$
$$-2xy^2 \quad +2 \quad +2 - 2xy^2$$

$$\frac{\cancel{2x^2y}\frac{dy}{dx}}{2x^2y} = \frac{2-2xy^2}{2x^2y} = \frac{2(1-xy^2)}{2x^2y}$$

$$\frac{dy}{dx} = \frac{1-xy^2}{x^2y}$$

$$\frac{d^2y}{dx^2} = \frac{\left[0 - 1\cdot y^2 + -x\cdot 2y\frac{dy}{dx}\right]x^2y - (1-xy^2)\left(2xy + x^2\cdot\frac{dy}{dx}\right)}{(x^2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left[-y^2 - 2xy\left(\frac{1-xy^2}{x^2y}\right)\right]x^2y - \left((1-xy^2)\left(2xy + \cancel{x}^2\cdot\frac{1-xy^2}{\cancel{x^2}y}\right)\right)}{x^4y^2}$$

$$\frac{d^2y}{dx} = \frac{-y^2\cdot x^2y - 2xy\left(\frac{1-xy^2}{\cancel{x^2y}}\right)\cancel{x^2y} - \left[2xy + \frac{1-xy^2}{y} - 2x^2y^3 - xy^2\frac{(1-xy^2)}{\cancel{x}}\right]}{x^4y}$$

48.

$$1 - xy = x - y$$

$$0 - 1\cdot y + -x\cdot\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$-1 \quad \cancel{+\frac{dx}{dx}} \quad -1 \quad + x\frac{dy}{dx}$$

$$-y - 1 = \frac{-dy}{dx} + x\frac{dy}{dx}$$

$$\frac{-y-1}{(-1+y)} = \frac{\frac{dy}{dx}\cancel{(-1+x)}}{\cancel{(-1+x)}}$$

$$\frac{-y-1}{-1+y} = \frac{dy}{dx} = \frac{-1(y+1)}{-1(1-x)} =$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(1-x) - (y+1)(-1)}{(1-x)^2} = \frac{\frac{y+1}{\cancel{(1-x)}}\cdot\cancel{(1-x)} + (y+1)}{(1-x)^2} = \frac{2y+2}{(1-x)^2}$$

50.

$$y^2 = 10x \Rightarrow y = \pm\sqrt{10x}$$

$$2y\frac{dy}{dx} = 10$$

$$\frac{dy}{dx} = \frac{10}{2y} = \frac{5}{y} = 5\cdot y^{-1}$$

$$\frac{d^2y}{dx} = -5y^{-2}\frac{dy}{dx} = \frac{-5}{y^2}\cdot\frac{5}{y} = \frac{-25}{y^3} = \frac{-25}{(\pm\sqrt{10x})^3} = \frac{\pm 25}{10x\sqrt{10x}}$$

$1. y = e^{\cos x}$

$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x)e^{\cos x} = -e^{\cos x} \cdot \sin x$

1. $\frac{d}{dx}[e^x] = e^x$

2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$

Example 3 Find y' if $y^2 + e^y = 2x^2$

$2y\frac{dy}{dx} + e^y \cdot \frac{dy}{dx} = 4x$

$\frac{dy}{dx}\cancel{(2y+e^y)} = 4x$

$\frac{\frac{dy}{dx}\cancel{(2y+e^y)}}{\cancel{2y+e^y}} = \frac{4x}{2y+e^y}$

$4. y = e^{\tan(5-7x^2)}$

$$\frac{dy}{dx} = e^{\tan(5-7x^2)} \cdot -14x \sec^2(5-7x^2)$$

$y = \tan(5-7x^2) \Rightarrow y = \tan u$

$u = 5-7x^2$

$\frac{du}{dx} = -14x$

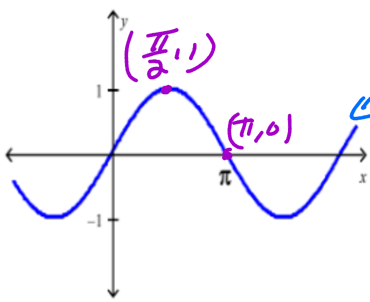
$\frac{dy}{du} = \sec^2 u$

$\frac{du}{dx} \cdot \frac{dy}{du}$

$-14x(\sec^2 u)$

$-14x(\sec^2(5-7x^2))$

$y = \sin x$



When you create an inverse you switch the x and the y.

$$x = \sin y$$

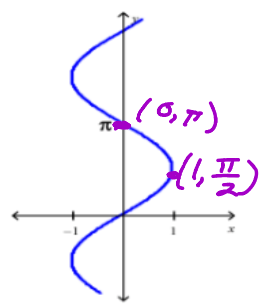
Then you solve for y.

$$y = \sin^{-1} x$$

$\sin y = \sin(\sin^{-1} x)$

$\sin y = x$

This is not a function! ☹



Definition of Inverse Trigonometric Function		
Function	*Domain*	*Range*
$y = arc\sin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = arc\cos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = arc\tan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = arc\cot x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = arc\sec x$ iff $\sec y = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = arc\csc x$ iff $\csc y = x$	$\lvert x\rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$

Example 1: Suppose $y = \sin^{-1} x$. Find $\frac{dy}{dx}$ using implicit differentiation

$\sin y = x$

$\frac{\cancel{\cos y}\frac{dy}{dx}}{\cancel{\cos y}} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \frac{\pm 1}{\sqrt{1-x^2}}$

$\sin^2 y + \cos^2 y = 1$

$x^2 + \cos^2 y = 1$

$\sqrt{\cos^2 y} = \sqrt{1-x^2}$

$\cos y = \pm\sqrt{1-x^2}$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

4) $y = cot^{-1} e^{2x}$

$\cot y = e^{2x}$

$$\frac{-\cancel{\csc^2 y}\cdot\frac{dy}{dx}}{\cancel{-\csc^2 y}} = \frac{e^{2x}\cdot 2}{-\csc^2 y}$$

$$\cot^2 y + 1 = \csc^2 y$$

$$\frac{\cos^2 y}{\sin^2 y} + \frac{\sin^2 y}{\sin^2 y}$$

$$\frac{\cos^2 y + \sin^2 y}{\sin^2 y} = \frac{1}{\sin^2 y}$$

$$\frac{dy}{dx} = \frac{2e^{2x}}{-\csc^2 y} = \frac{2e^{2x}}{-(\cot^2 y + 1)} = \frac{2e^{2x}}{-[(e^{2x})^2 + 1]} = \frac{-2e^{2x}}{e^{4x}+1}$$

Same

$y = \cot^{-1} e^{2x}$

$$\frac{d}{dx}[\cot^{-1} u] = \frac{-u'}{1+u^2} = \frac{-2e^{2x}}{1+(e^{2x})^2} = \frac{-2e^{2x}}{1+e^{4x}}$$

Function $f(x)$	Its Inverse $f^{-1}(x)$
$\{(2, 9), (4, 5), (11, 6)\}$	$\{(9, 2), (5, 4), (6, 11)\}$

Domain, Range

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What is:

- $f(2) = 9$
- $f(4) = 5$
- $f(11) = 6$

What is:

- $f^{-1}(9) = 2$
- $f^{-1}(5) = 4$
- $f^{-1}(6) = 11$

If given $f(2) = 4$ and $f(4) = 5$. What is $f^{-1}(4)$?

Derivative of an Inverse Function at (x, y)

$$(f^{-1})'(x) = \frac{1}{f'(y)}$$

The derivative of $f^{-1}(x)$ at the point (p,q) is the reciprocal derivative of $f(x)$ at (q,p)

$F'(x) = 3x^2$

Let $f(x) = x^3 - 1$ find $\left(f^{-1}\right)'(26)$. $=$ $(f^{-1})'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$

$F(x) = 26$

what does

$x = ?$

$3^3 - 1 = 27 - 1 = 26$

$$\frac{1}{F'(F^{-1}(26))} = \frac{1}{F'(3)} = \frac{1}{3(3^2)} = \frac{1}{27}$$

$F^{-1}(26) = ? \Rightarrow F(3) = 26 \Rightarrow F^{-1}(26) = 3$

Example 4

Let $f(x) = \frac{1}{4}x^3 + x - 1$. Find $(f^{-1})'(x)$ when $x = 3$.

$F'(x) = \frac{3}{4}x^2 + 1$

$(F^{-1})'(3)$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$\frac{1}{F'(F^{-1}(3))} = \frac{1}{F'(2)} = \frac{1}{\frac{3}{4}(2)^2+1}$

$\frac{1}{\frac{3}{\cancel{4}}\cancel{4}+1}$

$\frac{1}{3+1} = \frac{1}{4}$

$\frac{1}{4}x^3 + x - 1 = 3$

$\frac{1}{4}(2)^3 + 2 - 1$

$\frac{8}{4} + 2 - 1 = 2 + 2 - 1 = 3$

$F(2) = 3 \qquad F^{-1}(3) = 2$

a) $y = \sin^{-1}(x^2)$

$\sin y = x^2$

$\sin^2 y + \cos^2 y = 1$

$\cos y = \sqrt{1 - \sin^2 y}$

$\cos y \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{\sqrt{1-(x^2)^2}} = \frac{2x}{\sqrt{1-x^4}}$